

Seifert surfaces distinguished by sutured Floer homology but not its Euler characteristic

Faramarz Vafaee
Michigan State University
vafaee@msu.edu

12th April 2012

Abstract

In this paper we find a family of knots with trivial Alexander polynomial, and construct two non-isotopic Seifert surfaces for each member in our family. In order to distinguish the surfaces we study the sutured Floer homology invariants of the sutured manifolds obtained by cutting the knot complements along the Seifert surfaces. Our examples provide the first use of sutured Floer homology, and not merely its Euler characteristic (a classical torsion), to distinguish Seifert surfaces. Our technique uses a version of Floer homology, called "*longitude Floer homology*" in a way that enables us to bypass the computations related to the *SFH* of the complement of a Seifert surface.

1 Introduction

It is known that every knot in S^3 bounds a Seifert surface. Seifert surfaces play an important role in knot theory and low dimensional topology in general. The minimum genus taken over all oriented surfaces that a knot K bounds is called the genus of K . It is natural to wonder whether or not a given minimal genus Seifert surface for a knot is unique. To make sense of this question, we should be clear on the notion of equivalence between surfaces. We consider two surfaces R and R' to be equivalent if there is an isotopy of S^3 taking R to R' . Fiberedness of a knot is known as a sufficient condition for which its minimal genus Seifert surface is unique (see [BZ67]). However, there are many known examples of knots with non-isotopic Seifert surfaces. See for instance [Alf70, Alt11, Eis77, HJS08, Kak92, Kak05, Kob89, Lyo74, Tro75].

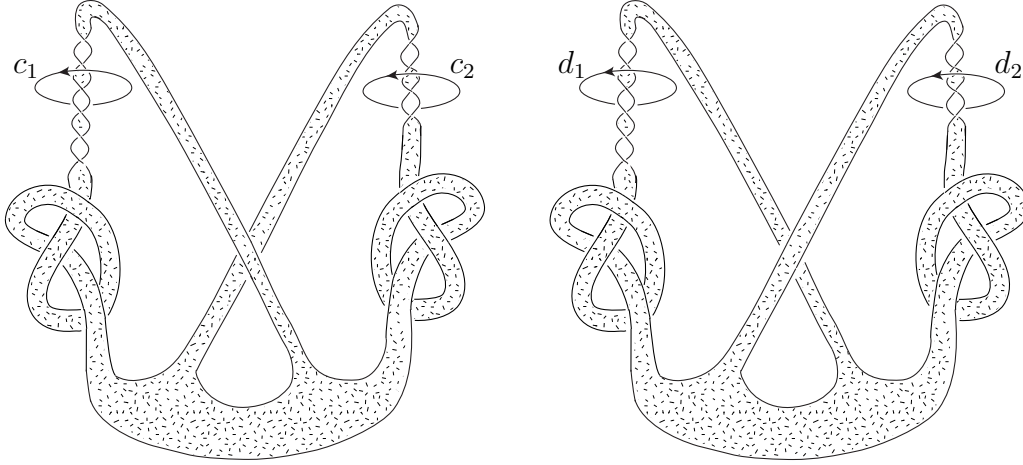


Figure 1: The above pictures are over/under plumbings of two twisted annuli, R and R' respectively, both are bounded by the same knot, $P(K_1, K_2)$, where K_1 is the right handed trefoil and K_2 is the left handed trefoil. These lead to two distinguished Seifert surfaces R and R' , up to weak equivalence, for the knot $P(K_1, K_2)$. The simple closed curves, c_1 , c_2 , d_1 and d_2 are basis elements for H_1 of the complement of these surfaces inside S^3 .

Many classical tools in distinguishing surfaces deal with the surfaces' complements in S^3 . These tools include, for instance, Seifert forms and the fundamental group of the surfaces' complements. They are quite powerful, but, they can potentially lead to tedious algebraic computations. There are also examples beyond the scope of classical tools.

In this paper we find knots with trivial Alexander polynomial and two distinguished Seifert surfaces for each. The idea is that we plumb two untwisted annuli. Then we tie arbitrary nontrivial knots, K_1 and K_2 in each of the annuli. We produce some twists in each annulus in such a way that the framings are l and 0 , respectively, where l is an arbitrary non-zero integer. We will see in section 3 that R and its dual, R' , both are bounded by the same knot $P(K_1, K_2)$. Figure 1 shows an example. Our main theorem shows that the two surfaces are inequivalent, provided that one of the twisting parameters is zero and the other is non-zero. The above strategy works regardless of the knots one ties to these annuli, i.e., if you tie arbitrary knots to the annuli, the two different plumbings result in the same knot.

We are now in a position to state the main theorem.

Theorem 1. *Let $P(K_1, K_2)$ be the knot obtained by plumbing two annuli with arbitrary knots K_1 and K_2 as in Figure 1, with framings l and 0 , respectively, $l \neq 0$. Changing the plumbing results in the same knot, but two inequivalent Seifert surfaces, R and R' .*

Our technique begins by noting that the surfaces' complements have a particular structure called a sutured manifold (see [Gab83a]). Assigned to sutured manifolds there is an invariant called sutured Floer homology (denoted by SFH) introduced by Juhász in [Juh06]. One cannot possibly use only the rank of SFH of minimal genus Seifert surfaces' complements to distinguish them, since the rank in this case depends only on the knot (see [Juh08, Theorem 1.5]). Therefore, we would need to know the structure of SFH as a Spin^c -graded group if we ever were able to use it to know two surfaces are not equivalent. Combining the sutured Floer homology of $(S^3(R), \gamma)$ with the Seifert form turns out to be a useful tool in distinguishing different Seifert surfaces (see [HJS08]). Altman in [Alt11] gives an example of using only the sutured Floer homology polytope to distinguish two Seifert surfaces for a knot (for related definitions see Section 2).

The reason we are interested in the particular knots here is twofold. First, the classical methods fail in distinguishing the two Seifert surfaces. Second, the polytopes of the surfaces' complements are the same. Our theorem produces the first examples where the $(SFH) + (\text{Seifert form})$ technique is successful, but $\chi(SFH)$ alone wouldn't have sufficed. Indeed, anytime the twisting of one of the annuli is zero, we have $\chi = 0$. We refer the reader to [Juh10] for a detailed discussion about the identification of the Euler characteristic of the sutured Floer homology with a type of Turaev torsion polynomial.

We close this section by mentioning that there are other notions of equivalence one could consider. The one we work with throughout the paper (so called *weak equivalence*) is the same as regarding two surfaces R and R' to be equivalent if there is an orientation preserving diffeomorphism between the pairs (S^3, R) and (S^3, R') (see [Hat83]). There is also a more restrictive notion called *strong equivalence*, that considers two Seifert surfaces for a knot K as equivalent if they are ambient isotopic to each other in $S^3 \setminus n(K)$, where $n(K)$ is a neighborhood of the knot K inside S^3 . While it was known that the Seifert surfaces we construct via over/under plumbings for the knots in our examples produce two distinguished Seifert surfaces for the knots up to strong equivalence (see [Gab86, Corollary 3.2]), what we show in the paper is stronger, that is, the two Seifert surfaces are not weakly equivalent. From now on, we will not make any further references to strong equivalence.

Acknowledgements

I would like to thank my advisor Matthew Hedden for all his invaluable support and instructive comments during the course of this work. I am also grateful to Eaman Eftekhary, Chuck Livingston, Luke Williams and David Krcatovic for helpful discussions and insights.

2 Background

Sutured manifolds were introduced by Gabai in [Gab83a]. Sutured Floer homology is a generalization of Ozsváth-Szabó Floer homology to an invariant of sutured manifolds, and is defined in [Juh06].

In this section we begin by briefly recalling some basic notions about sutured manifolds. We then discuss the structure of SFH as a group and how it behaves under decomposing a sutured manifold along an embedded surface. A key to understand it is a generalization of the Thurston norm to an invariant of relative homology classes in a sutured manifold called the Sutured-Thurston norm. We end this section by recalling a fact about the behavior of SFH under a Murasugi sum of two surfaces.

2.1 Sutured manifolds

The following with more details is contained in [Juh06].

Definition 2.1. *A sutured manifold (M, γ) is a compact oriented 3-manifold with boundary, together with a set $\gamma \subset \partial M$ consisting of annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a suture, i.e., a homologically non-trivial simple closed curve. The union of the sutures is denoted by $s(\gamma)$.*

We then take $R(\gamma) = \partial M \setminus \text{int}(\gamma)$. Define $R_+(\gamma)$ ($R_-(\gamma)$) to be those components of $R(\gamma)$ whose normal vector points out of (into) M . The orientation of $R(\gamma)$ must be coherent with respect to $s(\gamma)$, i.e., if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture. (M, γ) is called balanced if M has no closed components, $\chi(R_-(\gamma)) = \chi(R_+(\gamma))$, and the map from $\pi_0(A(\gamma))$ to $\pi_0(\partial M)$ is surjective.

Let $S^3(R_i) = S^3 \setminus \text{int}(R_i \times I)$ denote the complement of our Seifert surface. We equip this with a suture $\gamma = \partial R_i \times \{1/2\}$.

Definition 2.2. A sutured manifold (M, γ) is called taut if M is irreducible and $R(\gamma)$ is incompressible and Thurston norm minimizing in its homology class in $H_2(M, \gamma)$.

In general, for a Seifert surface R of a knot K it follows that $S^3(R)$ is taut if and only if $g(R) = g(K)$. In particular, both $S^3(R)$ and $S^3(R')$ in our examples are taut sutured manifolds.

2.2 Sutured Floer homology polytope and decomposition of a sutured manifold along an embedded surface

In this subsection we consider the SFH of a sutured manifold as a Spin^c -graded group. We recall the definition of the sutured Floer homology polytope and then describe how the shape of the polytope changes when one decomposes a sutured manifold along an embedded surface. Throughout, we use the notation of [Juh10]. We do not define all the terms here and refer the reader to [Juh06], [Juh08] and [Juh10] for related definitions and also more details.

Let (M, γ) be a balanced sutured manifold. Let also v_0 be a nowhere vanishing vector field pointing into M along $R_+(\gamma)$, pointing out of M along $R_-(\gamma)$ which restricts to γ to be the gradient of a height function $s(\gamma) \times I \rightarrow I$. The space of such vector fields is contractible. Thus, it makes sense to fix a representative, v_0 .

To a sutured manifold (M, γ) , one can assign a Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ where Σ is a compact oriented surface with boundary and $\boldsymbol{\alpha} = \{\alpha_1, \alpha_2, \dots, \alpha_d\}$ and $\boldsymbol{\beta} = \{\beta_1, \beta_2, \dots, \beta_d\}$ are two sets of pairwise disjoint simple closed curves in $\text{int}(\Sigma)$. Take $\mathbb{T}\boldsymbol{\alpha} = \alpha_1 \times \alpha_2 \times \dots \times \alpha_d$ and $\mathbb{T}\boldsymbol{\beta} = \beta_1 \times \beta_2 \times \dots \times \beta_d$ as subsets of the symplectic manifold $\text{Sym}^g(\Sigma)$ (see [Juh06] for details). To every point $x \in \mathbb{T}\boldsymbol{\alpha} \cap \mathbb{T}\boldsymbol{\beta}$, one can associate a relative Spin^c structure, $\mathfrak{s}(x) \in \underline{\text{Spin}}^c(M, \gamma)$ as follows:

First pick a Morse function which determines the Heegaard diagram whose gradient vector field agrees with v_0 along ∂M . Next, modify the vector field in a neighborhood of the flowlines specified by x . This produces a non-vanishing vector field v that agrees with v_0 on ∂M . The homology class of v specifies a relative Spin^c structure which we denote by $\mathfrak{s}(x)$. It turns out that $\underline{\text{Spin}}^c(M, \gamma)$ is an affine space over $H^2(M, \partial M; \mathbb{Z})$. Therefore,

it makes sense to talk about the difference of two relative Spin^c structure, $\mathfrak{s}(x) - \mathfrak{s}(y) \in H^2(M, \partial M; \mathbb{Z})$. We denote $PD^{-1}[\mathfrak{s}(x) - \mathfrak{s}(y)]$ by $\epsilon(x, y)$ which is an element of $H_1(M; \mathbb{Z})$.

Definition 2.3. *Let (M, γ) be a balanced sutured manifold. The support of sutured Floer homology of (M, γ) is*

$$S(M, \gamma) = \{\mathfrak{s} \in \text{Spin}^c(M, \gamma) : SFH(M, \gamma, \mathfrak{s}) \neq 0\}.$$

$SFH(M, \gamma)$ is a finitely generated abelian group (see [Juh06]), thus, $S(M, \gamma)$ is finite. Moreover, if (M, γ) is taut then $S(M, \gamma) \neq \emptyset$ by [Juh08, Theorem 1.4].

It turns out that v_0^\perp is a trivial vector bundle over ∂M , provided that (M, γ) is balanced in each component. Let us denote the set of all trivializations of v_0^\perp by $T(M, \gamma)$. For $\mathfrak{t} \in T(M, \gamma)$, let $c_1(\mathfrak{s}, \mathfrak{t}) \in H^2(M, \partial M; \mathbb{Z})$ be the relative Euler class of the vector bundle v^\perp with respect to the trivialization \mathfrak{t} , where v is a nowhere zero vector field along M which agrees with v_0 on ∂M . In other words, $c_1(\mathfrak{s}, \mathfrak{t})$ is the obstruction to extending \mathfrak{t} from ∂M to a trivialization of v^\perp over M .

Definition 2.4. *Fix $\mathfrak{t} \in T(M, \gamma)$. Define*

$$C(M, \gamma, \mathfrak{t}) = \{i(c_1(\mathfrak{s}, \mathfrak{t})) : \mathfrak{s} \in S(M, \gamma)\} \subset H^2(M, \partial M; \mathbb{R}),$$

where $i : H^2(M, \partial M; \mathbb{Z}) \rightarrow H^2(M, \partial M; \mathbb{R})$ is the map induced by the natural embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Let $P(M, \gamma, \mathfrak{t})$ be the polytope obtained as the convex hull of $C(M, \gamma, \mathfrak{t})$ inside $H^2(M, \gamma; \mathbb{R})$. Thus if (M, γ) is taut and $\alpha \in H^2(M, \partial M)$, then,

$$c(\alpha, \mathfrak{t}) = \min\{\langle c, \alpha \rangle : c \in C(M, \gamma, \mathfrak{t})\},$$

is a well-defined number.

Definition 2.5. *For $\alpha \in H^2(M, \partial M)$, let*

$$H_\alpha = \{x \in H^2(M, \partial M; \mathbb{R}) : \langle x, \alpha \rangle = c(\alpha, \mathfrak{t})\}.$$

In addition, we take

$$P_\alpha = H_\alpha \cap P(M, \gamma, \mathfrak{t}),$$

and we also use the notation,

$$SFH_\alpha(M, \gamma) = \bigoplus_{\{\mathfrak{s} \in \text{Spin}^c(M, \gamma) : i(c_1(\mathfrak{s}, \mathfrak{t})) \in P_\alpha(M, \gamma, \mathfrak{t})\}} SFH(M, \gamma, \mathfrak{s}).$$

It turns out that this is independent of \mathfrak{t} .

The following useful fact is contained in [Juh10, Proposition 4.13].

Proposition 2.6. *Let the sutured manifold (M, γ) be taut and strongly balanced. Fix an element $\alpha \in H_2(M, \partial M)$. Then $P_\alpha(M, \gamma, \mathfrak{t})$ is a face of the polytope $P(M, \gamma, \mathfrak{t})$. If S is a nice decomposing surface that results in a taut decomposition $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ and $[S] = \alpha$, then,*

$$SFH(M', \gamma') \cong SFH_\alpha(M, \gamma).$$

2.3 Sutured-Thurston norm and depth of a sutured manifold

We now recall the definitions of different norms assigned to a sutured manifold and discuss how they are related. All is contained in [FJR11].

Definition 2.7. *Let (M, γ) be a sutured manifold. Given a properly embedded, compact connected oriented surface $S \subset M$, let,*

$$x^s(S) = \max\{0, \frac{1}{2}|S \cap s(\gamma)| - \chi(S)\},$$

and extend this definition to disconnected surfaces by taking the sum over the components.

Now for $\alpha \in H_2(M, \partial M)$, take,

$$x^s(\alpha) = \min\{x^s(S) : [S, \partial S] = \alpha\}.$$

where the minimum is taken over all properly embedded surfaces $S \subset M$.

Example 2.8. *Let $K \subset S^3$ be a knot and let $(M, \gamma) = (S^3 \setminus n(K), \gamma)$ where $n(K)$ is a neighborhood of the knot K and γ consists of two meridional sutures. If $\alpha \in H_2(M, \partial M)$ is a generator, then $x^s(\alpha) = 2g(K)$. Note that this differs from the usual Thurston norm x of M , which satisfies $x(\alpha) = 2g(K) - 1$ for a nontrivial knot.*

Definition 2.9. Let $S(M, \gamma)$ be the support of $SFH(M, \gamma)$. If $\alpha \in H_2(M, \partial M; \mathbb{R})$, we define,

$$z(\alpha) = \max\{\langle \mathfrak{s} - \mathfrak{t}, \alpha \rangle : \mathfrak{s}, \mathfrak{t} \in S(M, \gamma)\}.$$

The following key proposition helps us to bypass the computations of the SFH of the complements of our annuli.

Proposition 2.10. [FJR11, Proposition 7.7] Let (M, γ) be an irreducible balanced sutured manifold such that all boundary components of M are tori. Then $z = x^s$.

We do not aim to give a precise proof of the proposition here; however, the proof uses a key fact which is helpful during the course of computations we will do. The detailed proof is contained in [FJR11, Proposition 7.7].

Sketch of the proof. It turns out that we can assume each of the components of ∂M consists of two sutures. For (M, γ) , there is a link, L , in a 3-manifold Y , where Y is obtained by Dehn filling ∂M such that the μ_i 's are meridians of the tori. For each $\mathfrak{s} \in \text{Spin}^c(M, \gamma)$ we obtain a relative first Chern class $c_1(\mathfrak{s}) \in H^2(M, \partial M)$ such that the set $\{c_1(\mathfrak{s}) : \mathfrak{s} \in S(M, \gamma)\}$ is symmetric about the origin. Then, due to [Juh08, Remark 8.5], for every $h \in H_2(M, \partial M)$,

$$\max\{\langle c_1(\mathfrak{s}), h \rangle : \mathfrak{s} \in S(M, \gamma)\} = x(h) + \sum_{i=1}^l |\langle h, \mu_i \rangle|. \quad (1)$$

Since the image of $S(M, \gamma)$ is symmetric and $\text{Spin}^c(M, \gamma)$ is an affine space over $H^2(M, \partial M)$, this is equivalent to saying

$$\max\{\langle \mathfrak{s} - \mathfrak{t}, h \rangle : \mathfrak{s}, \mathfrak{t} \in S(M, \gamma)\} = x(h) + \sum_{i=1}^l |\langle h, \mu_i \rangle|. \quad (2)$$

Note that the left sides of (1) and (2) must be the same which completes the proof. \square

2.4 SFH of the Murasugi sum of two manifolds

In [Juh06], Juhász found a formula that governs the behavior of SFH under a plumbing of two annuli. We recall [Juh06, Remark 10.8] as a proposition here.

Proposition 2.11. If a surface R is a Murasugi sum of two subsurfaces R_1 and R_2 , then over any field \mathbb{F} , we have,

$$SFH(S^3(R); \mathbb{F}) \cong SFH(S^3(R_1); \mathbb{F}) \otimes SFH(S^3(R_2); \mathbb{F}).$$

The above formula is an isomorphism of Spin^c -graded groups. Note also that a plumbing is a special case of a Murasugi sum (see also [Ni06]). As a matter of fact, what we have in Figure 1 is a Murasugi sum of two annuli along a 4-gon, so this proposition enables us to compute the SFH of the complement of each of those annuli and then simply take their tensor product.

3 Using $SFH(Y(R))$ to distinguish Seifert surfaces

This section will be devoted to using the sutured Floer homology invariants in order to distinguish inequivalent Seifert surfaces of knots in the form of Figure 1. We keep using the same notation as in Section 1. First, we show that, different plumbings of the knotted annuli have the same boundaries. Then, we will prove, the over/under plumbings lead to two distinguished Seifert surfaces for their common boundary.

3.1 R and R' bound the same knot.

The goal of this subsection is to prove that the knotted annuli in the form of Figure 1, where the right handed trefoil and the left handed trefoil are replaced by arbitrary knots, K_1 and K_2 , both are bounded by the same knot, $P(K_1, K_2)$. We would like to mention here that, it is possible to do isotopies in Figure 1 to go from one presentation of $P(K_1, K_2)$ to the other, however, we take a different route. Let us recall the definition of a plumbing here (see [Gab83b], [Gab86], [Kak05] and [Kob89] for more details). Given compact oriented surfaces $S_1, S_2 \subset S^3$; if there are 3-balls $V_1, V_2 \subset S^3$ satisfying the following properties:

$$\begin{aligned} V_1 \cup V_2 &= S^3, & V_1 \cap V_2 &= \partial V_1 = \partial V_2 = S^2, & S_i &\subset V_i \ (i = 1, 2), \\ R &= S_1 \cup S_2 \text{ and } & D &= S_1 \cap S_2 \text{ is a 4-gon,} \end{aligned}$$

then R is called a plumbing of S_1 and S_2 . In our examples S_1, S_2 and their plumbing are shown in Figure 2. Put $P(K_1, K_2) = \partial R$. Note that $R' = (R - D) \cup D'$ is an oriented surface with $\partial R' = P(K_1, K_2)$ where $D' = S^2 - \text{int}(D)$. We will say that R' is a *dual* of R . Notice that R' is also a plumbing of S_1 and S_2 where $S'_i = (S_i - D) \cup D'$ ($i = 1, 2$).

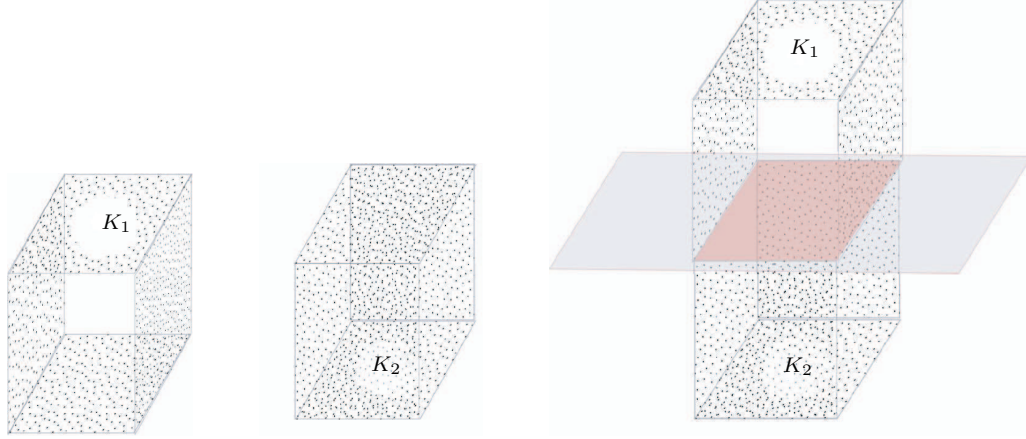


Figure 2: The first two pictures are showing two knotted annuli. The third picture shows they are plumbed along a red 4-gon where the 4-gon is on the sphere, one annulus is inside the sphere and the other annulus is outside of it. The light blue plane is part of the sphere.



Figure 3: The plumbing region, D' , is represented by the whole sphere where the 4-gon, D is removed from it. To have D' in between the two annuli, the top annulus will get moved down and the bottom annulus will go to the top.

We still need to argue that the two different plumbings we talked about above are the same as the plumbings in Figure 1. Notice that if we pull down the K_2 -knotted annulus in the left picture of Figure 1, we obtain the representation of a plumbing in the sense of Figure 2. Then, with the above notation, if we change the plumbing region, D to D' , we will have a hole in the sphere as in Figure 3. Now, in order to have the red plumbing region in Figure 3, between the knotted annuli, we need to push the K_1 -knotted annulus down, and the K_2 -knotted annulus will get moved up. But, the latter is the same as the right side of Figure 1.

Remark 3.1. *The above argument essentially shows that if we push the Seifert surfaces R and R' inside the four-ball and keep the knot, $P(K_1, K_2)$, in S^3 , they are isotopic. Pushing the plumbing disk, D , of the surface R inside the four-ball, enables us to go from the right picture in Figure 2 to the right picture in Figure 3 and simultaneously avoid all intersections that could possibly occur. Therefore, we obtain an isotopy between R and R' inside the four-ball.*

3.2 Classical Methods

In this subsection we observe that classical methods fail in distinguishing the two Seifert surfaces, at least for a subfamily of our examples. Let us start by looking at the Seifert forms. Pick that subset of our examples where the non-zero twisting parameter is 1. Easy computation based on the basis elements represented in Figure 1, shows that the Seifert forms are given by $V_R = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $V_{R'} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ for R and R' , respectively.

One can check that if $W = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$, then we have $V_{R'} = W^T V_R W$. Since, $W \in SL_2(\mathbb{Z})$, V_R and $V_{R'}$ are congruent. Thus, Seifert forms are incapable in distinguishing these particular surfaces. Another effective way to distinguish between surfaces is by looking at the homeomorphism type of their complements. Starting from the left picture in Figure 1, we take a regular neighborhood of the surface, R , inside S^3 , we obtain a handlebody with one zero handle and two one handles. The left picture in Figure 4 illustrates a schematic diagram for our discussion when we retract the handles to one zero cell and two one cells. Notice that we can do handle slides inside S^3 in the sense that in the schematic diagram, first, we obtain the middle picture in Figure 4 and then, the right picture, that is a planar diagram for the surface, R' . Hence, $S^3 \setminus R$ and $S^3 \setminus R'$ are homeomorphic. Thus any algebro-topological invariant derived from the homeomorphism type (e.g. π_1)

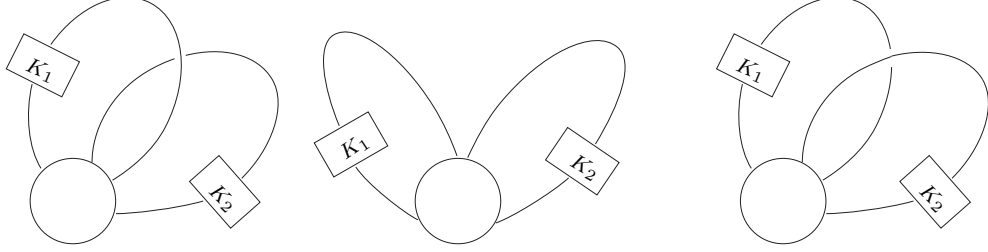


Figure 4: We encode the surfaces in this picture with planar diagrams where the circles and the bands are representing the zero cells and the one cells, respectively.

will fail to distinguish the surfaces R and R' .

3.3 Structure of $SFH(S^3(R))$

As we discussed in section 1, we plumb two knotted annuli. Figure 1 shows an example when the knots, K_1 and K_2 , are the left handed and right handed trefoils, respectively. The point is that the complement of each of these annuli in S^3 deformation retracts to the knot complement, regardless of what the framings of those knots are. Thus, we have $H_1(M_i) \cong \mathbb{Z}$ where $M_i = S^3(A(K_i))$ and $A(K_i)$'s are the annuli as in Figure 5 ($i = 1, 2$). Notice also that we are in a position to use Proposition 2.10. Therefore we have $x^s(A(K_i)) = z(A(K_i))$. We recall that

$$x^s(A(K_i)) = \max\{0, \frac{1}{2}|A(K_i) \cap s(\gamma)| - \chi(A(K_i))\}$$

and that $x^s(A(K_i)) = 2g - 1$, similar to the computation we did in Example 2.8; since $g \geq 1$, the breadth is at least 1. Thus we have the following polytopes,

$$\begin{array}{c} \overline{\begin{array}{ccccccc} & c_1 & & c_1 & & c_1 & \\ | & & | & & | & & | \\ G_1 & & G_2 & & \dots & & G_n \end{array}} \\ \\ \overline{\begin{array}{ccccccc} & c_2 & & c_2 & & c_2 & \\ | & & | & & | & & | \\ H_1 & & H_2 & & \dots & & H_m \end{array}} \end{array}$$

for $S^3(A(K_1))$ and $S^3(A(K_2))$, respectively, where G_1 , G_n , H_1 and H_m are all non-zero and also we have that $m, n \geq 2$. We now plumb the annuli, so,

$$\begin{array}{ccccccc}
G_1 \otimes H_1 & \xrightarrow{c_1} & G_2 \otimes H_1 & \xrightarrow{c_1} & \dots & \xrightarrow{c_1} & G_n \otimes H_1 \\
c_2 \downarrow & & c_2 \downarrow & & & & c_2 \downarrow \\
G_1 \otimes H_2 & \xrightarrow{c_1} & G_2 \otimes H_2 & \xrightarrow{c_1} & \dots & \xrightarrow{c_1} & G_n \otimes H_2 \\
c_2 \downarrow & & c_2 \downarrow & & & & c_2 \downarrow \\
\vdots & & \vdots & & & & \vdots \\
c_2 \downarrow & & c_2 \downarrow & & & & c_2 \downarrow \\
G_1 \otimes H_m & \xrightarrow{c_1} & G_2 \otimes H_m & \xrightarrow{c_1} & \dots & \xrightarrow{c_1} & G_n \otimes H_m
\end{array}$$

due to Proposition 2.11 we get a tensor product formula. Thus, one obtains the polytope shown above for the manifold $S^3(R)$ where c_1 and c_2 are the standard basis elements for $H_1(S^3(R); \mathbb{Z})$, specified in Figure 1. In addition, for $S^3(R')$, we follow the exact same process that obviously results in the same polytope, except c_1 and c_2 are replaced by d_1 and d_2 . In summary, we find that the polytopes of $SFH(S^3(R), \gamma)$ and $SFH(S^3(R'), \gamma')$ are rectangular and at least four of the vertices in each rectangle have non-zero groups sitting on them.

3.4 Calculation

In this subsection we present a way to calculate the groups $G_1 \otimes H_1$, $G_1 \otimes H_m$, $G_n \otimes H_1$ and $G_n \otimes H_m$ sitting on four of the vertices of the above polytope. We explain the computations needed to obtain G_1 and G_n . Then, H_1 and H_m could be calculated in a quite similar way. Throughout, we fix the knot K_1 and for the sake of simplicity of notation, we set $K = K_1$.

A priori we deal with the complement of a knotted annulus, with two oriented sutures on the two edges of it (see Figure 5). Let us denote this manifold by M . Then, M is homeomorphic to the knot complement. As a sutured manifold, however, it is different; the sutures are not meridional. They are rather like oriented longitudes.

Remark 3.2. *$SFH(M, \gamma)$ is isomorphic to an invariant of knots called "Longitude Floer Homology" of either the zero or l framed knot. As in the construction of knot Floer homology ([OS04, Ras03]) we first find a Heegaard diagram for the knot complement in S^3 . While in there we add the meridian of the knot to the set of β curves to obtain a Heegaard diagram for S^3 ; in the longitude Floer homology case, we add a longitude that results in a Hee-*

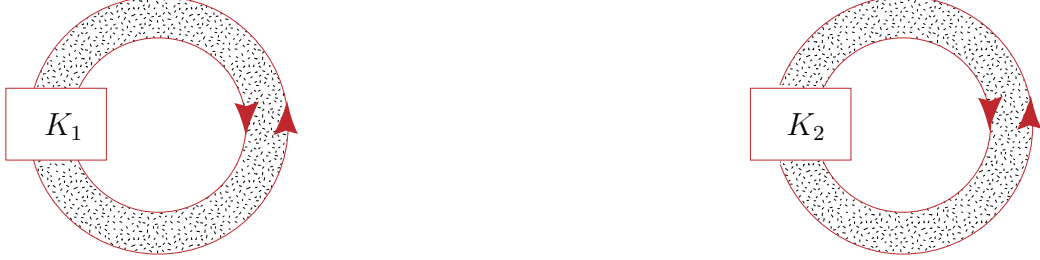


Figure 5: Knotted annuli with oriented sutures, $A(K_1)$ and $A(K_2)$

gaard diagram for $S_0^3(K)$. This subject has been first studied by Eftekhary in [Eft05] for framing zero and later, has been developed by Hedden for arbitrary surgery coefficients in [Hed07].

Let $S(K)$ be a Seifert surface for the knot K . The key to obtaining G_1 and G_n is to decompose (M, γ) along $S(K)$. The following proposition can be enlightening.

Proposition 3.3. *For a given knot K and its meridian μ , we have,*

$$\widehat{HFK}(S_N^3(K), \mu, g(K)) \cong \begin{cases} \widehat{HFK}(S^3, K, g(K)) & \text{if } N \neq 0 \\ \widehat{HFK}(S^3, K, g(K)) \oplus \widehat{HFK}(S^3, K, g(K)) & \text{if } N = 0 \end{cases}$$

Proof. Let (M, γ) be the knot complement with two oriented longitudes as the sutures (see Figure 5). We decompose (M, γ) along $S(K)$, a Seifert surface of K , to obtain (M', γ') ,

$$(M, \gamma) \rightsquigarrow^{S(K)} (M', \gamma').$$

Notice that,

$$(M', \gamma') \cong \begin{cases} S^3(S(K)) & \text{if } N \neq 0 \\ S^3(S(K)) \sqcup S^3 & \text{if } N = 0 \end{cases}$$

Now by taking SFH of both sides and also from the fact that $SFH(S^3(S(K))) \cong \widehat{HFK}(S^3, K, g(K))$, the result follows. The last isomorphism is [Juh08, Theorem 1.5]. \square

Proposition 3.3, in fact, gives us a formula for G_n . Note that $S(K)$ is a nice decomposing surface (see [Juh08] for the definition of a nice decomposing surface) and (M', γ') is taut. Therefore, based on Proposition 2.6, $SFH(M', \gamma')$ is a face of the polytope of $SFH(M, \gamma)$, that is, $SFH(M', \gamma') \cong G_n$. In order to obtain G_1 , we just need to decompose (M, γ) along the same surface, $S(K)$, with the opposite orientation. One can also obtain H_1 and H_m in a similar manner by replacing $S(K) = S(K_1)$ by $S(K_2)$. Thus, we have computed $G_1 \otimes H_1$, $G_1 \otimes H_m$, $G_n \otimes H_1$ and $G_n \otimes H_m$ in the polytope 3.4. In particular, these four groups are all non-zero since, $\widehat{HFK}(S^3, K_i, g(K_i)) \not\cong 0$, $i = 1, 2$, provided that K_i is nontrivial.

Proposition 3.3 can be interpreted in the language of *longitude Floer homology* as well. As we mentioned in Remark 3.2, the sutured Floer homology of the Seifert surface complementary manifold corresponding to the knot K , is isomorphic to $\widehat{HFL}(K)$. Therefore, when the framing is zero, we obtain the following corollary for \widehat{HFL} .

Corollary 3.4. *For a given knot K in S^3 ,*

$$\widehat{HFL}(K, \mathfrak{s}_{top}) \cong \widehat{HFK}(S^3, K, g(K)) \oplus \widehat{HFK}(S^3, K, g(K))$$

where \mathfrak{s}_{top} is the highest grading in the support of \widehat{HFL} , i.e., $\widehat{HFL}(K, i) \cong 0$ for $i > \mathfrak{s}_{top}$.

3.5 Proof of the main theorem

In this subsection we prove the main theorem of the paper. Having known the results of the subsections 3.3 and 3.4, it remains to distinguish R and R' . We mention here a natural notion of equivalence for sutured manifolds, where (M, γ) and (M', γ') are equivalent if $SFH(M, \gamma)$ and $SFH(M', \gamma')$ are isomorphic. We still need to clarify the notion of isomorphism between the sutured Floer homology of two sutured manifolds. The following is [HJS08, Definition 4.1].

Definition 3.5. *Two relatively $Spin^c$ -graded groups*

$$SFH(M, \gamma) = \bigoplus_{\mathfrak{s} \in \underline{Spin}^c(M, \gamma)} SFH(M, \gamma, \mathfrak{s}), \quad SFH(M', \gamma') = \bigoplus_{\mathfrak{s} \in \underline{Spin}^c(M', \gamma')} SFH(M', \gamma', \mathfrak{s})$$

are isomorphic if

1. *There is an isomorphism, $f^* : H^2(M', \partial M'; \mathbb{Z}) \rightarrow H^2(M, \partial M; \mathbb{Z})$.*

2. There is a bijection of sets $u : \underline{Spin}^c(M', \gamma') \rightarrow \underline{Spin}^c(M, \gamma)$.
3. The following diagram commutes

$$\begin{array}{ccc}
\underline{Spin}^c(M', \gamma') \otimes H^2(M', \partial M'; \mathbb{Z}) & \xrightarrow{(u, f^*)} & \underline{Spin}^c(M, \gamma) \otimes H^2(M, \partial M; \mathbb{Z}) \\
\downarrow & & \downarrow \\
\underline{Spin}^c(M', \gamma') & \xrightarrow{u} & \underline{Spin}^c(M, \gamma)
\end{array}$$

where the vertical arrows are followed by the action of $H^2(M', \gamma')$ on $\underline{Spin}^c(M', \gamma')$ and $H^2(M, \gamma)$ on $\underline{Spin}^c(M, \gamma)$.

4. There are isomorphisms $g_{\mathfrak{s}} : SFH(M', \gamma', \mathfrak{s}) \rightarrow SFH(M, \gamma, u(\mathfrak{s}))$ for every $\mathfrak{s} \in \underline{Spin}^c(M', \gamma')$

If $SFH(M, \gamma) \cong SFH(M', \gamma')$ in the sense of the above definition, f^* and u are both obtained by pulling back along a function $f : (M, \gamma) \rightarrow (M', \gamma')$; an equivalence between (M, γ) and (M', γ') . In addition, if the surfaces R and R' are weakly equivalent, then f comes from the restriction of $(S^3, R) \rightarrow (S^3, R')$ to $S^3(R)$ that gets mapped to $S^3(R')$. Also, $f_* : H_1(S^3 \setminus R) \rightarrow H_1(S^3 \setminus R')$ preserves the Seifert form, i.e. $a.b = f_*(a).f_*(b)$ for every $a, b \in H_1(S^3 \setminus R)$.

Proof of the main theorem. We recall that the Seifert matrices for the two surfaces R and R' are given by $V_R = \begin{pmatrix} l & 1 \\ 0 & 0 \end{pmatrix}$ and $V_{R'} = \begin{pmatrix} l & 0 \\ -1 & 0 \end{pmatrix}$ based on the basis elements c_i 's and d_i 's, shown in Figure 1 ($i = 1, 2$). Set $M = S^3(R)$ and $M' = S^3(R')$. Using the notation of Section 3, let us mark one generator in each $G_i \otimes H_j$ by x_{ij} and let us also denote the group generated by these elements, by $\langle x_{ij} \rangle$. If the two Seifert surfaces were equivalent, then based on Definition 3.5 we would have $\sigma : SFH(M, \gamma) \rightarrow SFH(M', \gamma')$, a bijection from the generators to generators, i.e., it maps every generator of $SFH(M, \gamma)$, say x_{ij} , to a generator $\sigma(x_{ij})$ of $SFH(M', \gamma')$. We also obtain a map $f : (M, \gamma) \rightarrow (M', \gamma')$, where f is compatible with taking difference classes, i.e., $\epsilon(a, b).\epsilon(c, d) = f_*\epsilon(a, b).f_*\epsilon(c, d)$ for some isomorphism $f_* : H_1(S^3 \setminus R) \rightarrow H_1(S^3 \setminus R')$ where, $a, b, c, d \in H_1(S^3 \setminus R)$. Suppose such a σ exists. As we mentioned earlier, in the Seifert forms V_R and $V_{R'}$, l is a non-zero integer. Let us assume that $l > 0$. Then,

$$\epsilon(\sigma(x_{11}), \sigma(x_{nm}))^2 = (f_*\epsilon(x_{11}, x_{nm}))^2 = (nc_1 + mc_2)^2 = n^2l + nm$$

Since σ sends generators to generators, for some α and β in \mathbb{Z} , $\epsilon(\sigma(x_{11}), \sigma(x_{nm})) = \alpha d_1 + \beta d_2$. The fact that f_* preserves the Seifert form implies that,

$$(\alpha d_1 + \beta d_2)^2 = n^2 l + nm.$$

On the other hand,

$$(\alpha d_1 + \beta d_2)^2 = \alpha^2 l - \alpha \beta.$$

Notice that x_{11} and x_{nm} have furthest distance in the polytope of $S^3(R)$. This implies that, $n^2 l + nm$ is the greatest positive number that can possibly be generated from $\alpha^2 l - \alpha \beta$, where $|\alpha| \leq n$ and $|\beta| \leq m$. Hence, we must have $|\alpha| = n$ and $|\beta| = m$. In addition, α and β have opposite signs. Thus, $\epsilon(\sigma(x_{11}), \sigma(x_{nm})) = \pm(nd_1 - md_2)$. The following illustrates what we just figured out,

$$\begin{array}{ccccccc} \langle x_{11} \rangle \xrightarrow{c_1} \dots \xrightarrow{c_1} \langle x_{k1} \rangle \xrightarrow{c_1} \dots \xrightarrow{c_1} \langle x_{1n} \rangle & & \langle \sigma(x_{ij}) \rangle \xrightarrow{d_1} \dots \xrightarrow{d_1} \langle \sigma(x_{k1}) \rangle \xrightarrow{d_1} \dots \xrightarrow{d_1} \langle \sigma(x_{11}) \rangle \\ c_2 \Big| & & c_2 \Big| & & d_2 \Big| & & d_2 \Big| \\ \vdots & & \vdots & & \vdots & & \vdots \\ c_2 \Big| & & c_2 \Big| & & d_2 \Big| & & d_2 \Big| \\ \langle x_{1m} \rangle \xrightarrow{c_1} \dots \xrightarrow{c_1} \langle x_{km} \rangle \xrightarrow{c_1} \dots \xrightarrow{c_1} \langle x_{nm} \rangle & & \langle \sigma(x_{nm}) \rangle \xrightarrow{d_1} \dots \xrightarrow{d_1} \langle \sigma(x_{km}) \rangle \xrightarrow{d_1} \dots \xrightarrow{d_1} \langle \sigma(x_{i'j'}) \rangle \end{array}$$

Observe that $\sigma(x_{11})$ and $\sigma(x_{nm})$ are located along the other diagonal compared to x_{11} and x_{nm} . Now, in the horizontal direction, take the closest nonzero group $G_k \otimes H_1$ to $\langle x_{11} \rangle$ such that $k > 0$. Such a group exists since, *a priori* $G_n \otimes H_1 \neq 0$. Thus, we can assume that $x_{k1} \neq 0$ for some $k > 0$. Based on our convention, $x_{k1} \in G_k \otimes H_1$ and $k \leq n$. Therefore, we have $\epsilon(x_{11}, x_{k1}) = \pm k c_1$. Then,

$$\epsilon(\sigma(x_{11}), \sigma(x_{k1}))^2 = (f_* \epsilon(x_{11}, x_{k1}))^2 = k^2 c_1^2 = k^2 l,$$

and for some α and β , $\epsilon(\sigma(x_{11}), \sigma(x_{k1})) = \alpha d_1 + \beta d_2$ which in turn ensures

$$(\alpha d_1 + \beta d_2)^2 = \alpha^2 l - \alpha \beta = k^2 l,$$

where $k > 0$, and so $\alpha \neq 0$. Since SFH is symmetric and we obtained the polytope from the tensor product formula and also from the way we chose x_{k1} , we conclude that α is at least k . The way that $\sigma(x_{11})$ is located in the polytope, implies that α and β have different signs. Thus, $\alpha \beta \leq 0$ and

$\alpha^2 - \alpha\beta \geq \alpha^2$, which implies $\beta = 0$ and $\epsilon(\sigma(x_{11}), \sigma(x_{k1})) = \pm kd_1$. We get a contradiction now. On the one hand,

$$\epsilon(\sigma(x_{11}), \sigma(x_{nm})).\epsilon(\sigma(x_{11}), \sigma(x_{k1})) = \pm(nd_1 - md_2). \pm kd_1 = \pm(nkl + mk),$$

where $l, m, n, k > 0$. On the other hand,

$$\epsilon(x_{11}, x_{nm}).\epsilon(x_{11}, x_{k1}) = \pm(nc_1 + mc_2). \pm kc_1 = \pm nkl.$$

For the case $l < 0$, instead of x_{11} , x_{nm} and x_{k1} we take x_{n1} , x_{1m} and $x_{(n-k)1}$ for the smallest positive k , where $x_{(n-k)1} \neq 0$. Then, a contradiction follows similarly. These show that $SFH(S^3(R)) \not\cong SFH(S^3(R'))$ and so $R \neq R'$. \square

References

- [Alf70] William Robert Alford. Complements of minimal spanning surfaces of knots are not unique. *Ann. of Math.*, 91(2):419–424, 1970.
- [Alt11] Irida Altman. Sutured floor homology distinguishes between seifert surfaces. *arXiv:1012.5904*, 3:1–15, 2011.
- [BZ67] Gerhard Burde and Heiner Zieschang. Neuwirthsche Knoten und Flächenabbildungen. *Abh. Math. Sem. Univ. Hamburg*, 31:236–249, 1967.
- [Eft05] Eaman Eftekhary. Longitude Floer homology and the Whitehead double. *Algeb. Geom. Topol.*, 5:1389–1418, October 2005.
- [Eis77] Julian R. Eisner. Knots with infinitely many spanning surfaces. *A.M.S.*, 229:329–349, 1977.
- [FJR11] Stefan Friedl, András Juhász, and Jacob Rasmussen. The decategorification of sutured Floer homology. *J. Topology*, 4(2):431–478, May 2011.
- [Gab83a] David Gabai. Foliations and the topology of 3-manifolds. *J. Differential Geom.*, 18:445–503, 1983.
- [Gab83b] David Gabai. The Murasugi sum is a natural geometric operation. *Contemporary Math. A.M.S.*, 20:131–143, 1983.
- [Gab86] David Gabai. Detecting fibred links in S^3 . *Comment. Math. Helv.*, 64(4):519–555, 1986.

- [Hat83] Allen E. Hatcher. A proof of the Smale Conjecture. *Ann. of Math.*, 117:553–607, 1983.
- [Hed07] Matthew Hedden. Knot Floer homology of Whitehead doubles. *Geom. Topol.*, 11:2277–2338, December 2007.
- [HJS08] Matthew Hedden, András Juhász, and Sucharit Sarkar. On sutured floer homology and the equivalence of seifert surfaces. *arXiv:0811.0178*, pages 1–32, 2008.
- [Juh06] András Juhász. Holomorphic discs and sutured manifolds. *Algeb. Geom. Topol.*, 6:1429–1457, October 2006.
- [Juh08] András Juhász. Floer homology and surface decompositions. *Geom. Topol.*, 12(1):299–350, March 2008.
- [Juh10] András Juhász. The sutured Floer homology polytope. *Geom. Topol.*, 14(3):1303–1354, May 2010.
- [Kak92] Osamu Kakimizu. Finding disjoint incompressible spanning surfaces for a link. *Hiroshima Math. J.*, 22:225–236, 1992.
- [Kak05] Osamu Kakimizu. Classification of the incompressible spanning surfaces for prime knots of 10 or less crossings. *Hiroshima Math. J.*, 35:47–92, 2005.
- [Kob89] Tsuyosht Kobayashi. Uniqueness of minimal genus Seifert surfaces for links. *Topology Appl.*, 33:265–279, 1989.
- [Lyo74] Herbert C Lyon. Simple knots without unique minimal surfaces. *A.M.S.*, 43(2):449–454, 1974.
- [Ni06] Yi Ni. Sutured Heegaard diagrams for knots. *Algeb. Geom. Topol.*, 6(1):513–537, April 2006.
- [OS04] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and knot invariants. *Adv. Math.*, 186(1):58–116, August 2004.
- [Ras03] Jacob Rasmussen. Floer homology and knot complements. *PhD Thesis, Harvard University*, 2003.
- [Tro75] H. F. Trotter. Some knots spanned by more than one knotted surface of minimal genus. *Princeton Univ. Press*, page 51.62, 1975.